

DIFFERENTIATING POLYNOMIALS, AND $\zeta(2)$

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ABSTRACT. We study the derivatives of polynomials with equally spaced zeros and find connections to the values of the Riemann zeta-function at the integers.

1. INTRODUCTION

Polynomials are fascinating because they have many facets to their personalities. By definition, a polynomial $f \in \mathbb{C}[x]$ is an expression of the form

$$(1.1) \quad f(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where the a_j are complex numbers. By the fundamental theorem of algebra, f also has a representation as a product

$$(1.2) \quad f(x) = a_n(x - x_1) \cdots (x - x_n),$$

where x_j are the roots of f . The fact that there are two ways of looking at polynomials provides possibilities that are hidden by their apparent simplicity.

In this paper we look at two different polynomials which have equally spaced zeros, and we study the zeros of their derivatives. We will see that Euler's famous result

$$(1.3) \quad \zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is lurking in the background, as are some generalizations of Euler's result. Our proof that $\zeta(2) = \pi^2/6$ appears to be new, and it doesn't use anything more than is typically covered in a second semester calculus course.

2. A POLYNOMIAL WITH ZEROS AT THE INTEGERS

The first polynomial we will consider is the degree $N + 1$ polynomial with simple zeros at $0, 1, 2, \dots, N$. That is,

$$(2.1) \quad p_N(x) := x(x - 1) \cdots (x - N) = \prod_{n=0}^N (x - n).$$

We will write $p(x)$ instead of $p_N(x)$ to simplify the notation. The question we will obtain a partial answer to is: Where are the zeros of the derivative $p'(x)$?

By Rolle's theorem, we know that p' has a zero in each of the intervals $(0, 1), (1, 2), \dots, (N - 1, N)$. Those zeros account for N of the zeros of p' , which is all of the zeros because p' has degree N .

We are left with the question: where in those intervals $(0, 1), \dots, (N - 1, N)$ are the zeros of p' ? Suppose more generally that f is a polynomial with only real zeros. There is a common

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misconception, reinforced by looking at small degree examples and badly drawn graphs, that the zeros of f' should be located close to the midpoint between neighboring zeros of f . A more accurate guide is the following:

Repulsion Principle. *Zeros of f' try to move away from the zeros of f .*

What the principle says is: when f' decides where to put a zero, it looks around to see what zeros of f are nearby. If it sees a bunch of zeros of f to the right, it will try to shift the zero of f' to the left, and vice-versa. This is illustrated in Figure 1, which shows part of the graph of a high-degree polynomial.

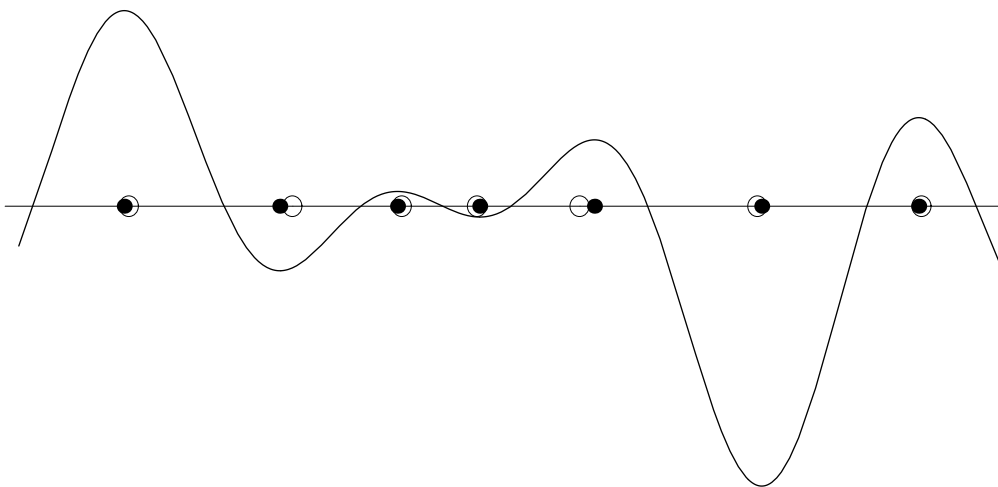


FIGURE 1. The circles are the midpoints of neighboring zeros of the polynomial f , and the dots are the zeros of f' . The zeros of f push the zeros of f' away from the midpoints.

Let's focus on the interval $(0, 1)$ for the polynomial $p(x)$ given above. Since there are many zeros to the right, and only one zero to the left, the Repulsion Principle says that p' will shift the zero to the left, so the zero will be closer to 0 than to 1. The question is: how close? We answer that question in the next section.

The Repulsion Principle, not by that name, is discussed extensively in our paper [1]. This principle explains, for example, the observation in [4] that the smallest zero of f' is closer to the smallest zero of f than to the next-smallest zero. The specific case of the polynomial p_N is discussed in [5].

3. FINDING ZEROS OF p'

Any attempt to find a simple formula for the location of the first zero of p' will fail for two reasons. First, there is no simple formula for the derivative p' . We discuss this difficulty in the next section. Second, there is no closed form expression for the zeros of an arbitrary high degree polynomial, so even if we had a nice formula for $p'(x)$ it might not do us any good.

3.1. A formula for $p'(x)$. To see the difficulty of finding a formula for $p'(x)$, first consider the product representation (2.1). To differentiate a product of $N + 1$ things, you have to use

the product rule N times. The result will be a sum of $N + 1$ terms, where each term is an N -fold product. The entire expression is quite unwieldy.

Since differentiating a sum is easier than differentiating a product, we can try multiplying out (2.1) to write $p(x)$ as a sum:

$$(3.1) \quad p(x) = \sum_{n=0}^{N+1} a_n x^n.$$

A few of the coefficients are easy to find. For example, $a_0 = 0$ because $p(0) = 0$, which we see by substituting $x = 0$ in the product. It is not too hard to see that $a_1 = (-1)^N N!$ and $a_{N+1} = 1$. A couple more coefficients can be described as simple formulas, but there is no known closed formula for all of the coefficients. These coefficients come up in a variety of situations and are known as the *signed Stirling numbers of the first kind*. We have $a_n = s(N + 1, n)$ in the usual notation for the Stirling numbers. Since, there is no simple formula for $s(N + 1, n)$, differentiating (3.1) does not give a useful expression. For more information about the Stirling numbers, see Chapter 6 of [2].

Since the product form of p is difficult to differentiate, and the sum form is easy to differentiate but has unusable coefficients, what can we do? The answer is: take logarithms! Since the log of a product is a sum, we have

$$(3.2) \quad \begin{aligned} \log(p(x)) &= \log \left(\prod_{n=0}^N (x - n) \right) \\ &= \sum_{n=0}^N \log(x - n). \end{aligned}$$

Note: here “log” means natural log, the function that calculator manufacturers and non-mathematicians call “ln”.

Differentiating the above expression we obtain “the logarithmic derivative” of p :

$$(3.3) \quad \frac{p'(x)}{p(x)} = \sum_{n=0}^N \frac{1}{x - n}.$$

This is useful because the solutions to $p'(x) = 0$ are the same as the solutions to

$$(3.4) \quad \frac{p'(x)}{p(x)} = 0,$$

except possibly at the points where $p(x) = 0$. However, we are trying to understand the solution to $p'(x) = 0$ which is between 0 and 1, so we have the following equivalent problem: find the solution to

$$(3.5) \quad \sum_{n=0}^N \frac{1}{x - n} = 0$$

which satisfies $0 < x < 1$. We will not find an exact expression for that solution, but we will find a good approximation.

3.2. First approximation to the solution. In this section we work informally and make things more precise in the following sections

Let α denote the solution to $p'(x) = 0$ with $0 < \alpha < 1$. As described in the previous section, we must solve

$$(3.6) \quad 0 = \frac{1}{\alpha} + \frac{1}{\alpha - 1} + \cdots + \frac{1}{\alpha - N}.$$

Since solving (3.6) is the same as finding a root of an N th degree polynomial, we should not expect to find an exact formula for α , except maybe for small N . Instead, we will look for an approximation.

Recall that the Repulsion Principle told us that α is close to zero, so in (3.6) the term $1/\alpha$ is much larger than the rest. So we will rearrange (3.6) to put $1/\alpha$ by itself:

$$(3.7) \quad \frac{1}{\alpha} = \frac{1}{1 - \alpha} + \frac{1}{2 - \alpha} + \cdots + \frac{1}{N - \alpha}.$$

Since α is close to 0, the right side of (3.7) will be close to the N th harmonic number:

$$(3.8) \quad H_N := 1 + \frac{1}{2} + \cdots + \frac{1}{N}.$$

We give more information about the harmonic numbers in the next section, but for now just recall that H_N is approximately $\log N$. Thus,

$$(3.9) \quad \frac{1}{\alpha} \approx \log N.$$

so, $\alpha \approx 1/\log N$, and we see that, for large N , α is actually quite close to 0. In particular, α is not close to the midpoint between the first two zeros.

The above calculation is informal and we do not intend to convey a precise meaning with the “approximately” symbol “ \approx ”. We will do the calculation rigorously in the next section.

4. BETTER APPROXIMATIONS TO THE FIRST ZERO OF p'

We make precise the argument in the previous section, and find that it touches on many combinatorial and number-theoretic objects that have been widely studied. In our approximation to the first root of $p'(x)$ we will encounter Euler’s constant γ and the Riemann zeta function

$$(4.1) \quad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Then in Section 5 we will differentiate another polynomial in order to evaluate $\zeta(2)$, and in Section 6 we evaluate the zeta function at the even integers. That calculation will involve the Bernoulli numbers, the Stirling numbers, and the Eulerian numbers.

To give a precise estimate for $0 < \alpha < 1$, the first root of p' , we need two facts. First, the geometric series

$$(4.2) \quad \frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^M + O(x^{M+1}),$$

which can be manipulated to give

$$(4.3) \quad \frac{1}{j - x} = \frac{1}{j} + \frac{x}{j^2} + \frac{x^2}{j^3} + \cdots + \frac{x^M}{j^{M+1}} + O\left(\frac{x^{M+1}}{j^{M+2}}\right),$$

as $x \rightarrow 0$. Second,

$$(4.4) \quad H_N := \sum_{n=1}^N \frac{1}{n} = \log(N) + \gamma + O\left(\frac{1}{N}\right).$$

Here γ is Euler's constant, $\gamma \approx 0.577$. Equation (4.4) is the *definition* of Euler's constant, but it appears in many other places, such as the Laurent series for the Riemann-zeta function:

$$(4.5) \quad \zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

There is a whole book on this interesting number [3].

We have

Theorem 4.1. *If $0 < \alpha < 1$ is a root of $p'_N(x)$ then*

$$(4.6) \quad \frac{1}{\alpha} = \log(N) + O(1),$$

so

$$(4.7) \quad \alpha = \frac{1}{\log(N)} + O\left(\frac{1}{\log^2(N)}\right).$$

Proof. The only fact we will use about α is that $0 < \alpha < 1$, so in particular $\alpha = O(1)$. Start with equation (3.7), and use equation (4.3) with $M = 0$, and (4.4):

$$\begin{aligned} \frac{1}{\alpha} &= \sum_{j=1}^N \frac{1}{j - \alpha} \\ &= \sum_{j=1}^N \left(\frac{1}{j} + O\left(\frac{\alpha}{j^2}\right) \right) \\ &= \sum_{j=1}^N \frac{1}{j} + O(\alpha) O\left(\sum_{j=1}^N \frac{1}{j^2}\right), \\ &= \log(N) + \gamma + O\left(\frac{1}{N}\right) + O(\alpha), \quad \text{since } \sum_n 1/n^2 \text{ converges} \\ &= \log(N) + O(1). \end{aligned}$$

That gives the first equation in the theorem. To obtain the second, take the reciprocal of both sides and use (4.3) with $j = \log N$ and $x = O(1)$. \square

We started with the assumption $0 < \alpha < 1$ and found an approximation for α with an error term $O(1/\log^2(N))$. By feeding this into an almost identical calculation, we can obtain an even better approximation to α .

Theorem 4.2. *If $0 < \alpha < 1$ is a root of $p'_N(x)$ then*

$$(4.8) \quad \frac{1}{\alpha} = \log(N) + \gamma + \frac{\zeta(2)}{\log(N)} + O\left(\frac{1}{\log^2(N)}\right),$$

so

$$(4.9) \quad \alpha = \frac{1}{\log(N)} - \frac{\gamma}{\log^2(N)} + \frac{\gamma - \zeta(2)}{\log^3(N)} + O\left(\frac{1}{\log^4(N)}\right).$$

Euler proved that

$$(4.10) \quad \zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In the proof below we will only use the fact that

$$(4.11) \quad \sum_{n=1}^N \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=N+1}^{\infty} \frac{1}{n^2} = \zeta(2) + O\left(\frac{1}{N}\right),$$

which follows from

$$(4.12) \quad \frac{1}{N+1} = \int_{N+1}^{\infty} \frac{1}{x^2} dx < \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \int_N^{\infty} \frac{1}{x^2} dx = \frac{1}{N}.$$

Proof of Theorem 4.2. Again we start with equation (3.7) and use equation (4.3), this time with $M = 1$, and (4.4). We have

$$\begin{aligned} \frac{1}{\alpha} &= \sum_{j=1}^N \frac{1}{j - \alpha} \\ &= \sum_{j=1}^N \left(\frac{1}{j} + \frac{\alpha}{j^2} + O\left(\frac{\alpha^2}{j^3}\right) \right) \\ &= \sum_{j=1}^N \frac{1}{j} + \alpha \sum_{j=1}^N \frac{1}{j^2} + O(\alpha^2) \sum_{j=1}^N O\left(\frac{1}{j^3}\right) \\ &= \log(N) + \gamma + O\left(\frac{1}{N}\right) + \alpha \left(\zeta(2) + O\left(\frac{1}{N}\right) \right) + O(\alpha^2), \end{aligned}$$

The last step used the fact that $\sum 1/n^3$ converges. On the final line above, replace α by $1/\log N + O(1/\log^2 N)$ and $O(\alpha^2)$ by $O(1/\log^2 N)$, both of which follow from Theorem 4.1. Since $1/N = O(1/\log N)$ this establishes the first formula in Theorem 4.2. The second formula follows in exactly the same way as the second formula in Theorem 4.1. \square

It is clear that we could use the result in Theorem 4.2, feed it back into equation (3.7), and obtain an even more precise formula for α . Repeating this process would establish that

$$(4.13) \quad \alpha = \frac{1}{\log(N)} + \frac{c_2}{\log^2(N)} + \cdots + \frac{c_M}{\log^M(N)} + O_M\left(\frac{1}{\log^{M+1}(N)}\right),$$

where the c_j are polynomial expressions in γ and $\zeta(2), \dots, \zeta(j-1)$. For the first few c_j we have

$$\begin{aligned} c_2 &= -\gamma \\ c_3 &= \gamma^2 - \zeta(2) \\ c_4 &= \zeta(3) - \gamma\zeta(2) + \zeta(2) - \gamma. \end{aligned}$$

In the next section we use a different polynomial, but which also has equally spaced zeros, to prove Euler's result $\zeta(2) = \pi^2/6$.

5. THE POLYNOMIALS $z^N - 1$, AND $\zeta(2)$

The previous section we considered polynomials with equally spaced zeros on the real line. In this section we consider polynomials with equally spaced zeros on the unit circle. That is, $q(z) = q_N(z) := z^N - 1$. If we set $e(z) := e^{2\pi iz}$ then the zeros of $q(z)$ are at the N th roots of unity: $1, e(1/N), e(2/N), \dots, e((N-1)/N)$. Thus,

$$(5.1) \quad z^N - 1 = \prod_{j \bmod N} (z - e(j/N)).$$

Here $j \bmod N$ refers to a set of representatives of the integers modulo N . This is legitimate because $e(z)$ has period 1. Later we will use the specific choice $-N/2 < j \leq N/2$.

We now use the polynomial $q(z)$ to evaluate $\zeta(2)$. As in the previous section, we begin with the logarithmic derivative of q . Since we are trying to prove an identity, in this case we will make use of both the product and the sum representation of the polynomial. Taking the *second* derivative of the logarithm of (5.1) we obtain

$$(5.2) \quad \frac{-N(N-1)z^{N-2} - Nz^{2N-2}}{(z^N - 1)^2} = - \sum_{j \bmod N} \frac{1}{(z - e(j/N))^2}.$$

Various formulas can be obtained by plugging in specific quantities for z in the above equation. It is tempting to plug in $z = 1$, which is possible if one first moves the $j \equiv 0 \bmod N$ term to the left side and simplifies. Equivalently, start over with the polynomial $(z^N - 1)/(z - 1)$. Instead, we will plug in $z = e(1/2N)$, a $2N$ th root of unity, which is motivated by noting that the left side will become fairly simple.

Multiplying both sides of that equation by z^2 and plugging in $z = e(1/2N)$ we have

Lemma 5.1. *We have*

$$(5.3) \quad \begin{aligned} \frac{N^2 - 2N}{4} &= -e(1/N) \sum_{j \bmod N} \frac{1}{(e(1/2N) - e(j/N))^2} \\ &= - \sum_{j \bmod N} \frac{1}{(1 - e((2j-1)/2N))^2}. \end{aligned}$$

The remainder of the calculation closely parallels the treatment in the previous section: we want to expand the denominator of the right side of (5.3) as a series. Using the Taylor series for e^x one can check that for $0 < |x| < 1$ we have

$$(5.4) \quad \frac{1}{(1 - e(x))^2} = -\frac{1}{4\pi^2 x^2} + \frac{i}{2\pi x} + \frac{5}{12} - \frac{1}{6}i\pi x + O(x^2).$$

The restriction on x is due to the fact that $e(x) = 1$ for integral x . We will only need the first term in that expansion.

Since (5.4) is only valid for $0 < |x| < 1$ we are forced to make the specific choice $-N/2 < j \leq N/2$ for the representatives of the integers modulo N . Making this choice and then

applying the Taylor expansion (5.4) we obtain

$$\begin{aligned} \frac{N^2 - 2N}{4} &= - \sum_{-N/2 < j \leq N/2} \frac{1}{(1 - e((2j-1)/2N))^2} \\ &= \sum_{-N/2 < j \leq N/2} \left(\frac{(2N)^2}{4\pi^2(2j-1)^2} + O\left(\frac{N}{|2j+1|}\right) \right) \\ &= \frac{N^2}{\pi^2} \sum_{-N/2 < j \leq N/2} \frac{1}{(2j-1)^2} + O(N \log N). \end{aligned}$$

Dividing both sides by N^2 and rearranging, we have

$$(5.5) \quad \sum_{-N/2 < j \leq N/2} \frac{1}{(2j-1)^2} = \frac{\pi^2}{4} + O\left(\frac{\log N}{N}\right).$$

Letting $N \rightarrow \infty$ and replacing $j \rightarrow 1-j$ for $j \leq 0$, we obtain

$$(5.6) \quad \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{\pi^2}{8}.$$

This is equivalent to Euler's result $\zeta(2) = \pi^2/6$ because letting

$$(5.7) \quad \zeta(k) = \sum_{j=1}^{\infty} \frac{1}{j^k} \quad \text{and} \quad \zeta_{\text{odd}}(k) = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^k}$$

we have the following lemma.

Lemma 5.2. *For $k > 1$,*

$$(5.8) \quad \zeta(k) = \frac{2^k}{2^k - 1} \zeta_{\text{odd}}(k).$$

Proof.

$$\begin{aligned} \zeta(k) &= \sum_{n=1}^{\infty} \frac{1}{n^k} = \sum_{n \text{ odd}} \frac{1}{n^k} + \sum_{n \text{ even}} \frac{1}{n^k} \\ &= \zeta_{\text{odd}}(k) + \sum_{n=1}^{\infty} \frac{1}{(2n)^k} \\ &= \zeta_{\text{odd}}(k) + \frac{1}{2^k} \zeta(k). \end{aligned}$$

Now solve for $\zeta(k)$. □

Therefore by (5.6),

$$(5.9) \quad \zeta(2) = \frac{4}{3} \zeta_{\text{odd}}(2) = \frac{\pi^2}{6}.$$

Note that substituting $z = e(t/N)$, $0 < t < 1$, into (5.2) and applying the same manipulations as above gives

$$(5.10) \quad \sum_{n \in \mathbb{Z}} \frac{1}{(n+t)^2} = \frac{\pi^2}{\sin^2(\pi t)}.$$

From this one can produce results analogous to Euler's, but where the sum is over an arithmetic progression.

In the next section we generalize the method to evaluate the zeta-function at the positive even integers.

6. ZETA AT THE EVEN INTEGERS

The same calculations as above with a little more book keeping allow us to compute $\zeta(2n)$ for all positive integers n . This general result was known to Euler and there are many proofs of it. To state the result we need to introduce the Bernoulli numbers. The Bernoulli numbers B_k for $k \geq 1$ are defined as the coefficients of the Taylor series for

$$(6.1) \quad \frac{z}{e^z - 1} = \sum_{k \geq 1} \frac{B_k}{k!} z^k = 1 - \frac{1}{2}z + \frac{1}{6} \frac{z^2}{2!} - \frac{1}{30} \frac{z^3}{3!} + \cdots.$$

For more about the Bernoulli numbers, see [2]. We can now state the general theorem which gives the values for $\zeta(2n)$.

Theorem 6.1. *For $n \geq 1$,*

$$(6.2) \quad \zeta(2n) := \sum_{k \geq 1} \frac{1}{k^{2n}} = \frac{(-1)^{n+1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}.$$

We provide a proof of Theorem 6.1 by differentiating the polynomials $z^N - 1$ in two different ways. In each case we will have an expression involving recursively defined constants that are well known combinatorial objects. Using a couple of identities for these numbers and letting N tend to infinity we will be able to evaluate $\zeta(2n)$.

6.1. Repeatedly “Differentiating”. As in the calculation of $\zeta(2)$, we will see that taking the logarithm of our polynomial before differentiating will allow us to express things in a useful way. However, unlike that calculation we will not simply take the derivative. Instead of repeatedly applying the operator $\frac{d}{dz}$ we apply the operator $z \frac{d}{dz}$. We will refer to this as “differentiation,” where the quotes indicate that we are applying $z \frac{d}{dz}$ instead of $\frac{d}{dz}$.

We wish to apply $z \frac{d}{dz}$ many times to the function

$$(6.3) \quad \log(z^N - 1) = \sum_{j \bmod N} \log \left(z - e \left(\frac{j}{N} \right) \right).$$

To understand high “derivatives” of the right hand side we will consider the high “derivatives” of the function $\log(z - a)$, where a is a complex number. We have:

Proposition 6.2. *Define $b_{k,m}$ by $b_{1,1} = 1$, and $b_{0,m} = 0$ for all $m \geq 0$, and*

$$(6.4) \quad b_{k,m+1} = k b_{k,m} - (k-1) b_{k-1,m}$$

for $m \geq 1$ and $0 \leq k \leq m$. We have

$$(6.5) \quad \left(z \frac{d}{dz} \right)^m \log(z - a) = \sum_{k=1}^m b_{k,m} z^k (z - a)^{-k}.$$

Proof. Begin by noticing that

$$(6.6) \quad \left(z \frac{d}{dz}\right) (z^k (z-a)^{-k}) = k (z^k (z-a)^{-k} - z^{k+1} (z-a)^{-k-1})$$

for $k \geq 1$. We will now establish the result by induction. It is true for $m = 1$. Assuming it is true for all $r < m$, with $m \geq 2$, we have

$$\begin{aligned} \left(z \frac{d}{dz}\right)^m \sum_{j \bmod N} \log(z-a) &= \left(z \frac{d}{dz}\right) \sum_{k=1}^{m-1} b_{k,m-1} z^k (z-a)^{-k} \\ &= \sum_{k=1}^{m-1} b_{k,m-1} (k z^k (z-a)^{-k} - k z^{k+1} (z-a)^{-k-1}) \\ &= \sum_{k=1}^m (k b_{k,m-1} - (k-1) b_{k-1,m-1}) z^k (z-a)^{-k} \\ &= \sum_{k=1}^m b_{k,m} z^k (z-a)^{-k}. \end{aligned}$$

The first step follows by the induction hypothesis and the final step follows by the definition of the $b_{k,m}$. \square

Using this result we see that

$$\begin{aligned} \left(z \frac{d}{dz}\right)^m \sum_{j \bmod N} \log(z - e(j/N)) &= \sum_{j \bmod N} \sum_{k=1}^m b_{k,m} z^k (z - e(j/N))^{-k} \\ (6.7) \quad &= \sum_{k=1}^m b_{k,m} \sum_{j \bmod N} z^k (z - e(j/N))^{-k}, \end{aligned}$$

which is an expression for the “derivatives” of the right side of (6.3). Next we will find an expression for the derivatives of the left side.

Proposition 6.3. Define $c_{k,m}$ by $c_{1,1} = 1$, $c_{k,1} = 0$ for all $k > 1$, and $c_{0,m} = 0$ for all $m \geq 0$, and

$$(6.8) \quad c_{k,m+1} = -(k c_{k,m} + (m+1-k) c_{k-1,m})$$

for $k \geq 1$ and $m \geq 1$. We have

$$(6.9) \quad \left(z \frac{d}{dz}\right)^m \log(z^N - 1) = \frac{N^m \sum_{k=1}^m c_{k,m} z^{Nk}}{(z^N - 1)^m}.$$

Proof. Define $Q_m(z) := (z^N - 1)^m \left(z \frac{d}{dz}\right)^m \log(z^N - 1)$. An easy inductive argument shows that $Q_m(z)$ is a polynomial. In the notation of the Proposition we wish to prove that

$$(6.10) \quad Q_m(z) = N^m \sum_{k=1}^m c_{k,m} z^{Nk}.$$

We will obtain a recursion for Q_m and then using that recursion we will find a recursion for its coefficients, $c_{k,m}$.

By definition we have

$$\begin{aligned}\frac{Q_{m+1}(z)}{(z^N - 1)^{m+1}} &= \left(z \frac{d}{dz} \right) \frac{Q_m(z)}{(z^N - 1)^m} \\ &= \frac{z(z^N - 1)Q'_m(z) - mNz^N Q_m(z)}{(z^N - 1)^{m+1}}.\end{aligned}$$

Thus we obtain the recursion

$$\begin{aligned}Q_{m+1}(z) &= z(z^N - 1)Q'_m(z) - mNz^N Q_m(z) \\ &= z^{N+1}Q'_m(z) - zQ'_m(z) - mNz^N Q_m(z).\end{aligned}$$

By induction we have the following

$$\begin{aligned}z^{N+1}Q'_m(z) &= N^{m+1} \sum_{k=1}^m k c_{k,m} z^{N(k+1)} \\ zQ'_m(z) &= N^{m+1} \sum_{k=1}^m k c_{k,m} z^{Nk} \\ mNz^N Q_m(z) &= N^{m+1} \sum_{k=1}^m m c_{k,m} z^{N(k+1)}\end{aligned}$$

Combining these we obtain the recursive formula for the constants, $c_{k,m}$. □

6.2. Combinatorial Numbers. In Propositions 6.2 and 6.3 we introduced two sets of numbers $\{b_{k,m}\}$ and $\{c_{k,m}\}$, and those numbers satisfied particular recursion formulas. It turns out that those numbers are related to other famous numbers from combinatorics: Stirling numbers of the second kind and the Eulerian numbers.

The Stirling numbers of the second kind $S(n, k)$ count the number of ways to partition a set of n elements into k nonempty subsets. They can be described recursively by $S(1, 1) = 1$ and the recursion

$$(6.11) \quad S(m, k) = S(m-1, k-1) + kS(m-1, k).$$

On the other hand, the Eulerian number $e_{k,m}$ is the number of permutations of the numbers 1 to m in which exactly $k+1$ elements are greater than the previous element. The Eulerian numbers are given recursively by $e_{1,1} = 1$ and $e_{k,1} = 0$ for $k > 1$, and

$$(6.12) \quad e_{k,m} = (m-k+1)e_{k-1,m-1} + ke_{k,m-1}.$$

It is the recursive definitions which we will make use of.

We have the following:

Proposition 6.4. *In the notation above,*

$$\begin{aligned}b_{k,m} &= (m-1)!S(m, k) \\ c_{k,m} &= (-1)^{m-1}e_{k,m}.\end{aligned}$$

In particular, $b_{m,m} = (-1)^{m-1}(m-1)!$.

Proof. We will not prove the first of these two results because the only fact we need about $b_{n,m}$ is $b_{m,m} = (-1)^{m-1}(m-1)!$. To prove this, it is enough to note that by the recursion in Proposition 6.2 we have

$$(6.13) \quad b_{m,m} = mb_{m,m-1} - (m-1)b_{m-1,m-1} = -(m-1)b_{m-1,m-1}.$$

Thus the result follows by induction and the fact that $b_{1,1} = 1$.

By the definition of $c_{k,m}$ in Proposition 6.3 we see that $c_{k,m}$ and $(-1)^{m-1}e_{k,m-1}$ satisfy the same recurrence relation with the same initial conditions. \square

The following relates the Eulerian numbers that appear in our formulas to the Bernoulli numbers which appear in the formula for $\zeta(2n)$. This Lemma is Exercise 72 in Chapter 6 of [2].

Lemma 6.5.

$$(6.14) \quad \sum_{k=1}^n (-1)^k c_{k,n} = (-1)^n 2^n (2^n - 1) \frac{B_n}{n}.$$

6.3. Limit as $N \rightarrow \infty$. We now return to our discussion of the two expressions for the “derivatives” of $\log(z^N - 1)$. As in the calculation of $\zeta(2)$, to evaluate $\zeta(2n)$ we begin by equating the two expressions for $(z \frac{d}{dz})^{(2n)} \log(z^N - 1)$ given by equation (6.7) and Proposition 6.3. Using the Taylor series expansion that we used in the previous section and similar analysis we are led to the evaluation of $\zeta_{\text{odd}}(2n)$.

Proof of Theorem 6.1. Set $m = 2n$. Beginning with equation (6.7) and setting $z = e(1/2N)$ we have

$$\begin{aligned} \left(z \frac{d}{dz}\right)^m \sum_{j \bmod N} \log(z - e(j/N)) \Big|_{z=e(1/2N)} &= \sum_{k=1}^m b_{k,m} \sum_{j \bmod N} e(1/2N)^k (e(1/2N) - e(j/N))^{-k} \\ &= \sum_{k=1}^m b_{k,m} \sum_{j \bmod N} \left(1 - e\left(\frac{(2j-1)}{2N}\right)\right)^{-k} \end{aligned}$$

As in the calculation from Section 5 we will use the Taylor series

$$(6.15) \quad (1 - e(x))^{-1} = \frac{-1}{2\pi i x} + O(1),$$

which is valid for $0 < |x| < 1$. To use the Taylor series we must have $-N/2 < j \leq N/2$ and so we obtain

$$\begin{aligned}
& \sum_{k=1}^m b_{k,m} \sum_{-N/2 < j \leq N/2} \left(1 - e \left(\frac{(2j-1)}{2N} \right) \right)^{-k} \\
&= \sum_{k=1}^m b_{k,m} \sum_{-N/2 < j \leq N/2} \left[\left(\frac{2N}{2\pi i(2j-1)} \right)^k + O \left(\frac{N^{k-1}}{(2j-1)^{k-1}} \right) \right] \\
&= b_{m,m} \sum_{-N/2 < j \leq N/2} \left(\frac{2N}{2\pi i(2j-1)} \right)^m + O_m(N^{m-1}) \\
&\quad + \sum_{k=2}^{m-1} b_{k,m} \sum_{-N/2 < j \leq N/2} \left[\left(\frac{2N}{2\pi i(2j-1)} \right)^k + O_k \left(\frac{N^{k-1}}{(2j-1)^{k-1}} \right) \right] \\
&\quad + \sum_{-N/2 < j \leq N/2} \left[\left(\frac{2N}{2\pi i(2j-1)} \right) + O(1) \right] \\
(6.16) \quad &= b_{m,m} \left(\frac{N}{\pi i} \right)^m \sum_{-N/2 < j \leq N/2} \left(\frac{1}{(2j-1)} \right)^m + O_m(N^{m-1}) + O(N \log(N)).
\end{aligned}$$

We write O_m because the implied constant depends on m . The $O(N \log N)$ term is only necessary when $m = 2$.

Next we will turn to our other expression for $(z \frac{d}{dz}) \log(z^N - 1)$, which was given in Proposition 6.3. As before, evaluating at $e(1/2N)$ we have

$$\begin{aligned}
(6.17) \quad & \frac{N^m \sum_{k=1}^m c_{k,m} e(1/2N)^{Nk}}{(e(1/2N)^N - 1)^m} = \frac{N^m}{(-2)^m} \sum_{k=1}^m c_{k,m} (-1)^k \\
&= \frac{N^m}{2^m} \left((-1)^m 2^m (2^m - 1) \frac{B_m}{m} \right) \\
&= N^m \left((2^m - 1) \frac{B_m}{m} \right),
\end{aligned}$$

where the second equality follows from the Lemma 6.5 and we use that m is even. By equation (6.3) we have that equations (6.16) and (6.17) are equal. Setting the last lines of each equation equal and dividing by N^m we have

$$(6.18) \quad \left((2^m - 1) \frac{B_m}{m} \right) = b_{m,m} (\pi i)^{-m} \sum_{-N/2 < j \leq N/2} \left(\frac{1}{(2j-1)} \right)^m + O_m(N^{-1}) + O(N^{1-m} \log(N)).$$

Letting N go to infinity we obtain

$$(6.19) \quad \frac{b_{m,m}}{(\pi i)^m} \sum_{j=-\infty}^{\infty} \frac{1}{(2j-1)^m} = \frac{(2^m - 1)B_m}{m}.$$

To finish the proof we use the facts that $b_{m,m} = (-1)^{m-1}(m-1)!$ from Lemma 6.2 and Lemma 5.2 to obtain

$$\begin{aligned}\zeta(m) &= \frac{2^m \zeta_{\text{odd}}(m)}{(1-2^m)} = \frac{-2^{m-1} \pi^m i^m B_m}{m!} \\ &= \frac{2^m}{(1-2^m)} \frac{1}{2} \sum_{j=-\infty}^{\infty} \frac{1}{(2j-1)^m} \\ &= \frac{2^{m-1}}{(1-2^m)} \frac{(2^m-1)(\pi i)^m B_m}{m!}.\end{aligned}$$

Using $m = 2n$ we have

$$(6.20) \quad \zeta(2n) = \frac{\pi^{2n} (-1)^{n-1} 2^{2n-1} B_{2n}}{(2n)!},$$

the desired result. □

Instead of the substitution $z = e(1/2N)$ one can set $z = e(A/BN)$ to obtain an expression for

$$(6.21) \quad \sum_{n \in \mathbb{Z}} \frac{1}{(A + Bn)^k}.$$

This can then be used to evaluate Dirichlet L-functions $L(s, \chi)$, with character χ , at the positive even (odd) integers if χ is even (odd).

REFERENCES

- [1] D.W. Farmer and R. Rhoades, *Differentiation evens out zero spacing*, Trans. Amer. Math. Soc., Vol 37, No. 9, 2005, p3789-3811. math.NT/0310252
- [2] R. Graham, D. Knuth, and O. Patashnik, “Concrete Mathematics”, Addison-Wesley, 1988.
- [3] J. Havil, “Gamma: Exploring Euler’s Constant”, Princeton University Press, 2003.
- [4] A. Melman, *Bounds on the Zeros of the Derivative of a Polynomial with All Real Zeros*, Amer. Math. Monthly, February 2008.
- [5] D. Romik, *Roots of the Derivative of a Polynomial*, Amer. Math. Monthly, January 2005.